Interview 2

Question 1

a) Pictured below is a circle with radius 1cm inscribing an equilateral triangle.

What is the area of the shaded region?

b) Assume that this pattern repeats indefinitely (each circle inscribes an equilateral triangle, and each equilateral triangle inscribes a circle).

What is the area of the shaded region?

Question 2

In this question, a *string* is defined as a sequence of symbols a and b. For example, ab, $abbbb$, and $baabbababba$ are all valid strings.

We define *constructable strings* as strings that can be constructed according to the following rules:

The string ab is a constructable string. New constructable strings can be formed by taking an existing constructable string and applying one of the following rules (where x represents any arbitrary string):

- Rule 1: $ax \xrightarrow{1} axx$ (If ax is a constructable string, then so is axx)
- Rule 2: $abbbx \xrightarrow{2} ax$ (If $abbbx$ is a constructable string, then so is ax)

For example, if ab is a constructable string, then abb is also a constructable string by applying Rule 1 (where $x = b$). This is written as $ab \xrightarrow{1} abb$.

Since we have shown that abb is a constructable string, then $abbbb$ is also a constructable string ($abb \xrightarrow{1} abbbb$, where $x = bb$). The full derivation would look like: $ab \xrightarrow{1} abb \xrightarrow{1} abbbb$.

If $abbbb$ is a constructable string, then we can apply Rule 2 to show that ab is a constructable string by applying Rule 2 ($abbbb \xrightarrow{2} ab$). This is a redundant operation, as our definition already defines ab as a constructable string.

From now on we will use the following shorthand: $a^2 = aa$, $a^3b = aaab$, etc.

Answer the following questions:

a) Is the string ab^5 constructable? If not, informally justify why there is no way to construct this string.

b) Is the string ab^3 constructable? If not, informally justify why there is no way to construct this string.

c) Prove that all strings $u = ab^n$ are constructable, where $n = 2^k - 3m \geqslant 0$ for some integers $k, m \geqslant 0$.

d) Prove (by induction) that if a string u can be constructed, then $u = ab^n$ where $n = 2^k - 3m \geqslant 0$ for some for some integers $k, m \geqslant 0$.

   Hint: Instead of induction over numbers, can you do induction over the provided rules? Using the string ab as the base case, and Rules 1 and 2 as the inductive cases.

Question 3

Sketch the graph:

$$y = \frac{\sin e^x}{e^x}$$

Pay close attention to:

- The coordinates of the y-axis intercept
- The function's behaviour as x approaches $\pm\infty$
- The equations of the function's asymptote(s).